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A short introduction to quantum symmetry

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In quantum theory, symmetries more general than groups are possible. We give a general definition of a quantum symmetry, such that symmetry operations act on the Hilbert space $\mathcal{H}$ of physical states and notions of unitarity, invariance and covariance are defined. Within this frame, weak quasi quantum groups are described as a natural generalization of group algebras. Consistency with locality distinguishes them from more general quantum symmetries. To find the new kinds of symmetry one should investigate low dimensional quantum systems such as two-dimensional layers.

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In classical mechanics, symmetries are groups of transformations acting on a phase space. Soon after the discovery of quantum mechanics, group symmetries turned out to be an important tool to obtain predictions of quantum theory. Recently, the study of low dimensional quantum field theories revealed signs of more general quantum symmetries ["(quasi) quantum group" symmetries] [1]. We will see that all these quantum symmetries can act on the Hilbert space of physical states and allow for a conventional interpretation as symmetry in quantum mechanics [2].

1. Quantum symmetry in quantum theory

Let us begin with a short review on group symmetries in quantum mechanics. Consider some quantum mechanical system $(\mathcal{H}, \{\Psi\}, |0\rangle, H)$ within a second quantized formalism. The Hilbert space $\mathcal{H}$ of physical states should contain a unique ground state $|0\rangle$ with respect to the Hamiltonian H. We assume that $\mathcal{H}$ is generated from $|0\rangle$ by multiplets of field operators $\Psi_i^I(\boldsymbol{r}, t)$ (I labels multiplets, i labels fields in the multiplet I) that create particles or excitations.

A compact group G is called a (internal) symmetry of this system if there is a unitary representation $\mathcal{U} : G \to \mathcal{B}(\mathcal{H})$ such that the ground state $|0\rangle$ and

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the Hamiltonian H are invariant and field operators Ψ_i^I transform covariantly according to the representation τ^I of G. To state these requirements in mathematical terms, let us recall two notions from the representation theory of groups. Every group has a trivial representation $\epsilon_G : G \to \mathbb{C}$ defined by $\epsilon_G(\xi) = 1$ for all $\xi \in G$. The tensor product $\boxtimes$ of representations τ, τ' is given by

$$(\tau \boxtimes \tau')_{kl,ij}(\xi) = \tau_{ki}(\xi)\tau'_{lj}(\xi) \quad \text{for all } \xi \in G. \tag{1.1}$$

If we set $\xi^* = \xi^{-1}$, unitarity of $\mathcal{U}$ asserts $\mathcal{U}(\xi)^* = \mathcal{U}(\xi)^{-1} = \mathcal{U}(\xi^{-1}) = \mathcal{U}(\xi^*)$. Invariance of the ground state $|0\rangle$ can be expressed as

$$\mathcal{U}(\xi)|0\rangle = |0\rangle = |0\rangle\epsilon_G(\xi) \quad \text{for all } \xi \in G. \tag{1.2}$$

We say that $\Psi_i^I(\boldsymbol{r},t)$ transforms covariantly according to the representation τ^I of G, if for all $\xi \in G$

$$\mathcal{U}(\xi)\Psi_i^I(\boldsymbol{r},t) = \Psi_j^I(\boldsymbol{r},t)\tau^I_{ji}(\xi)\mathcal{U}(\xi) = \Psi_j^I(\boldsymbol{r},t)(\tau^I \boxtimes \mathcal{U})_{ji}(\xi)\,. \tag{1.3}$$

Since we concentrate on internal symmetries, there is no action on the space–time variable of the field. For this reason we will often neglect to write the arguments $(\boldsymbol{r},t)$ explicitly.

In conclusion, the formulation of symmetry in quantum theory involves a conjugation $*$ to express unitarity, a trivial representation ϵ to state invariance and a tensor product of representations to write down the covariance law. The mathematical structure behind these notions is known as "(non-co-associative) bi-$*$-algebra" $(\mathcal{G}^*, \Delta, \epsilon, *)$. In detail this means that $\mathcal{G}^*$ is a $*$-algebra with unit e and $\Delta : \mathcal{G}^* \to \mathcal{G}^* \otimes \mathcal{G}^*$ (*co-product*), $\epsilon : \mathcal{G}^* \to \mathbb{C}$ (*co-unit*) are $*$-homomorphisms. For Δ the notion of a $*$-homomorphism involves a definition of $*$ on $\mathcal{G}^* \otimes \mathcal{G}^*$, which is not unique (cf. ref. [2]). Finally, the co-product Δ and the co-unit ϵ satisfy

$$(\epsilon \otimes \mathrm{id})\Delta = \mathrm{id} = (\mathrm{id} \otimes \epsilon)\Delta\,. \tag{1.4}$$

In fact, the co-product Δ determines a tensor product $\tau \boxtimes \tau'$ for representations τ, τ' of $\mathcal{G}^*$,

$$(\tau \boxtimes \tau')(\xi) = (\tau \otimes \tau')(\Delta(\xi)) \quad \text{for all } \xi \in \mathcal{G}^*. \tag{1.5}$$

With respect to this tensor product of representations, the co-unit ϵ furnishes a trivial one-dimensional representation. Triviality refers to the property $\epsilon \boxtimes \tau = \tau = \tau \boxtimes \epsilon$ for all representations τ of $\mathcal{G}^*$, which follows from (1.4).

Let us explain how to abstract a bi-$*$-algebra from the representation theory of the compact group G. In this case, $\mathcal{G}^*$ should denote the group algebra of the compact gauge group G, i.e. the space of "linear combinations" of elements in G. All homomorphisms of the group G can be uniquely extended to algebra homomorphisms of the group algebra $\mathcal{G}^*$. Consequently it suffices to fix $\Delta_G, \epsilon_G, *$ on elements ξ in the group G. $\epsilon_G, *$ have been defined above and comparison of (1.5) with (1.1) yields $\Delta_G(\xi) = \xi \otimes \xi$. Assuming that the action of $*$ on $\mathcal{G}^* \otimes \mathcal{G}^*$ is specified by $(\xi \otimes \eta)^* = \xi^* \otimes \eta^*$, $(\mathcal{G}^*, \Delta_G, \epsilon_G, *)$ is easily shown to satisfy all assumptions listed above. In this sense, groups are only special examples of bi-$*$-algebras.

Definition 1 ([2]). A (non-co-associative) bi-$*$-algebra $(\mathcal{G}^*, \Delta, \epsilon, *)$ is called *quantum symmetry* of the system $(\mathcal{H}, \{\Psi\}, |0\rangle, H)$, if there exists a representation

$$\mathcal{U} : \mathcal{G}^* \to \mathcal{B}(\mathcal{H})$$

such that

(i) $\mathcal{U}$ is unitary in the sense that $\mathcal{U}(\xi^*) = \mathcal{U}(\xi)^*$ for all $\xi \in \mathcal{G}^*$;

(ii) the Hamiltonian H and the ground state $|0\rangle$ are invariant, i.e.,

$$[H, \mathcal{U}(\xi)] = 0, \quad \mathcal{U}(\xi)|0\rangle = |0\rangle\epsilon(\xi) \quad \text{for all } \xi \in \mathcal{G}^*; \tag{1.6}$$

(iii) the field operators $\Psi_i^I(\boldsymbol{r}, t)$ transform covariantly with respect to the representation τ^I of $\mathcal{G}^*$, i.e.,

$$\begin{aligned} \mathcal{U}(\xi)\Psi_i^I(\boldsymbol{r}, t) &= \sum_p \Psi_j^I(\boldsymbol{r}, t)(\tau^I \boxtimes \mathcal{U})_{ji}(\xi) \qquad (1.7) \\ &= \sum_p \Psi_j^I(\boldsymbol{r}, t)\tau_{ji}^I(\xi_p^1)\mathcal{U}(\xi_p^2), \quad \text{if } \Delta(\xi) = \sum_p \xi_p^1 \otimes \xi_p^2. \end{aligned}$$

We mention that the transformation law of adjoint fields Ψ^* involves an antipode $\mathcal{S}$. Consequently, the existence of $\mathcal{S}$ should also be stated among the defining features of a quantum symmetry (cf. ref. [2]).

The covariant transformation law (1.7) tells us how to shift representation operators $\mathcal{U}(\xi)$ through fields from left to right. Together with the invariance of the ground state $|0\rangle$ it determines the transformation properties of states. We demonstrate this for the one-excitation states,

$$\begin{aligned} \mathcal{U}(\xi)\Psi_i^I|0\rangle &= \sum \Psi_j^I \tau_{ji}^I(\xi_p^1)\mathcal{U}(\xi_p^2)|0\rangle \\ &= \sum \Psi_j^I \tau_{ji}^I(\xi_p^1)\epsilon(\xi_p^2)|0\rangle = \Psi_j^I|0\rangle\tau_{ji}^I(\xi). \end{aligned}$$

The transformation law of higher excitations can be found along the same lines.

2. Weak quasi quantum groups

The examples of bi-$*$-algebras which come with compact groups have distinguished properties. We recall that the tensor product of groups is associative and commutative. This can be traced back to properties of the co-product Δ_G. In the last section, Δ_G was defined by $\Delta_G(\xi) = \xi \otimes \xi$ for all $\xi \in G$. Hence it is *co-associative*, i.e. $(\Delta_G \otimes \mathrm{id})\Delta_G = (\mathrm{id} \otimes \Delta_G)\Delta_G$, and *co-commutative* in the sense that $\Delta_G = \Delta'_G \equiv \sigma\Delta_G$, where $\sigma : \mathcal{G}^* \otimes \mathcal{G}^* \to \mathcal{G}^* \otimes \mathcal{G}^*$ is defined by $\sigma(\xi \otimes \eta) = \eta \otimes \xi$.

In this section we explain how co-associativity and co-commutativity can be weakened to allow for tensor products which are associative and commutative only up to equivalence. Let us emphasize that we do not assume $\Delta(e) = e \otimes e$. In general, $\Delta(e)$ will be some projector $P \in \mathcal{G}^* \otimes \mathcal{G}^*$ so that we can accommodate for truncated tensor products, i.e., the tensor product of two representations possibly vanishes on a nontrivial subspace of the full representation space.

The co-product Δ is called *quasi co-associative* if there is $\varphi \in \mathcal{G}^* \otimes \mathcal{G}^* \otimes \mathcal{G}^*$ and quasi inverse φ^{-1} such that

$$\varphi\varphi^{-1} = (\mathrm{id} \otimes \Delta)\Delta(e), \quad \varphi^{-1}\varphi = (\Delta \otimes \mathrm{id})\Delta(e), \tag{2.1}$$

$$\varphi(\Delta \otimes \mathrm{id})\Delta(\xi) = (\mathrm{id} \otimes \Delta)\Delta(\xi)\varphi \quad \text{for all } \xi \in \mathcal{G}^*. \tag{2.2}$$

This in turn implies that $(\pi^I \boxtimes \pi^J) \boxtimes \pi^K$ and $\pi^I \boxtimes (\pi^J \boxtimes \pi^K)$ are equivalent representations (but not equal). Following Drinfel'd [3] we postulate

$$(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\varphi)(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\varphi) = (e \otimes \varphi)(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\varphi)(\varphi \otimes e), \tag{2.3}$$

$$(\mathrm{id} \otimes \mathrm{id} \otimes \epsilon)(\varphi) = \Delta(e). \tag{2.4}$$

The tuple $(\mathcal{G}^*, \Delta, \epsilon, \varphi)$ is *quasi triangular*, if there is $R \in \mathcal{G}^* \otimes \mathcal{G}^*$ and quasi inverse R^{-1} such that

$$RR^{-1} = \Delta'(e), \qquad R^{-1}R = \Delta(e), \tag{2.5}$$

$$R\Delta(\xi) = \Delta'(\xi)R \quad \text{for all } \xi \in \mathcal{G}^*. \tag{2.6}$$

If $\Delta(\xi) = \sum_p \xi_p^1 \otimes \xi_p^2$ then $\Delta'(\xi) = \sum_p \xi_p^2 \otimes \xi_p^1$ by definition. Quasi triangularity implies $\pi^I \boxtimes \pi^J \cong \pi^J \boxtimes \pi^I$. We postulate the following two relations:

$$(\mathrm{id} \otimes \Delta)(R) = \varphi_{231}^{-1} R_{13} \varphi_{213} R_{12} \varphi^{-1}, \tag{2.7}$$

$$(\Delta \otimes \mathrm{id})(R) = \varphi_{312} R_{13} \varphi_{132}^{-1} R_{23} \varphi. \tag{2.8}$$

We used the standard notation. If $R = \sum r_a^1 \otimes r_a^2$ then $R_{13} = \sum r_a^1 \otimes e \otimes r_a^2$ etc. If s is any permutation of 123 and $\varphi = \sum \varphi_\sigma^1 \otimes \varphi_\sigma^2 \otimes \varphi_\sigma^3$ then

$$\varphi_{s(1)s(2)s(3)} = \sum_\sigma \varphi_\sigma^{s^{-1}(1)} \otimes \varphi_\sigma^{s^{-1}(2)} \otimes \varphi_\sigma^{s^{-1}(3)}. \tag{2.9}$$

These relations imply the validity of quasi Yang–Baxter equations,

$$R_{12}\varphi_{312}R_{13}\varphi_{132}^{-1}R_{23}\varphi = \varphi_{321}R_{23}\varphi_{231}^{-1}R_{13}\varphi_{213}R_{12}. \tag{2.10}$$

In the group situation we see that $\Delta(e) = e \otimes e = R, \varphi = e \otimes e \otimes e$ leading to the commutative and associative tensor product of representations. Quantum groups are obtained from groups by dropping the restriction $R = e \otimes e$. For Drinfel'd's quasi quantum groups [3] we keep only $\Delta(e) = e \otimes e$. In a last step of generalization we give up $\Delta(e) = e \otimes e$ to get weak quasi quantum groups. There exist nontrivial examples for all these structures.

It can be shown that all algebraic structures described in this section determine a representation of the braid group [4]. This has been applied to construct knot invariants from quasi quantum groups [5].

3. Quantum symmetry, locality and statistics

In quantum field theory, permutation group statistics is implemented through quadratic relations among the field operators, namely canonical (anti-)commutation relations for bosons (fermions). The spin statistics theorem states that fermions have spin $s = \frac{1}{2}, \frac{3}{2}, \ldots$, whereas bosons have integer spin. More general values for the spin (remember that the spin labels representations of the rotation group, e.g. SO(2) in two space dimensions) are possible in low dimensional quantum field theory. They are associated with braid group statistics. It has been proposed to implement braid group statistics through local braid relations [6],

$$\Psi_i^I(x,t)\Psi_j^J(y,t) = \Psi_l^J(y,t)\Psi_k^I(x,t)\mathcal{R}_{kl,ij}^{IJ>}\omega^{IJ}\ , \quad x > y\ . \tag{3.1}$$

Here $x > y$ refers to some ordering prescription (cf. ref. [6]) and ω^{IJ} should be complex numbers. In contrast to ref. [6] we do not restrict the $\mathcal{R}$-matrix to have $\mathbb{C}$-number entries, but the matrix elements should take values in $\mathcal{U}(\mathcal{G}^*)$ instead. Note that for $\mathcal{R} = 1$ and $\omega^{IJ} = \pm 1$ we recover Bose/Fermi commutation relations as a special case of eq. (3.1).

In a theory with weak quasi quantum group symmetry, consistency of (3.1) with the quantum symmetry $(\mathcal{G}^*, \Delta, \epsilon, *)$ and locality may be exploited to obtain constraints on the coefficients $\mathcal{R}_{kl,ij}^{IJ>}$. The analysis suggests

$$\mathcal{R}_{kl,ij}^{IJ>} = (\tau^I \otimes \tau^J \otimes \mathcal{U})_{kl,ij}(\varphi_{213}(R \otimes e)\varphi^{-1})\ . \tag{3.2}$$

To gain some insight into the structure of (3.2) we demonstrate that *local braid relations* (3.1) *with* $\mathcal{R}$ *given by* (3.2) *are at least consistent with the transformation law of fields*, i.e., that both sides of the equation transform in the same way. The products of covariant fields which appear in (3.1) are in general not covariant. However, one may use φ to construct a "covariant product" $\times$ of field operators [2,4],

$$(\Psi^I \times \Psi^J)_{ij} \equiv \sum \Psi_m^I \Psi_n^J \tau_{mi}^I(\varphi_\sigma^1)\tau_{nj}^J(\varphi_\sigma^2)\mathcal{U}(\varphi_\sigma^3)\ . \tag{3.3}$$

By (2.2), $\Psi^I \times \Psi^J$ *transforms covariantly according to the tensor product representation* $\tau^I \boxtimes \tau^J$. If this covariant product is used to rewrite the local braid relations (3.1), (3.2), consistency with the transformation law is evident from the intertwining properties of R, eq. (2.6).

One can also prove consistency of (3.1), (3.2) with associativity of the product of field operators. The calculations use eq. (2.3) and the quasi Yang–Baxter equations (2.10) [2].

Once (3.2) is established, it describes a direct connection between the the physical $\mathcal{R}$-matrix in (3.1) and the weak quasi quantum group structure of the quantum symmetry. Given R, φ one may use (3.2) to calculate $\mathcal{R}$. For a generalized quantum symmetry (i.e., $R \neq e \otimes e, \varphi \neq e \otimes e \otimes e$), $\mathcal{R}$ is always nontrivial, i.e., $\mathcal{R} \neq 1$. This explains why these symmetries have never been observed in higher dimensional quantum systems, where the Bose/Fermi alternative holds (i.e., $\mathcal{R} = 1$). In lower dimensional quantum field theory, the situation is different. We expect braid statistics of particles or excitations with non(half-)integer spin to be implemented by local braid relations (3.1), $\mathcal{R} \neq 1$, which are not consistent with group symmetries, since $R = e \otimes e$, $\varphi = e \otimes e \otimes e$ implies $\mathcal{R} = 1$. Thus we are forced to consider more general quantum symmetries.

Let us finally mention that there is an example of a local quantum field theory with nontrivial weak quasi quantum group symmetry [7]. In this model, covariant fields obey local braid relations (3.1), (3.2) with a nonnumerical $\mathcal{R}$-matrix.

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